Representation-Faithful Aggregation

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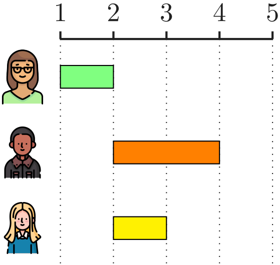
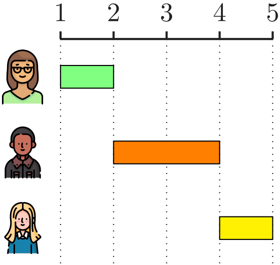
Aggregation across disciplines Workshop · December 16, 2021

Choosing the time slot for a meeting

Three agents have preferences over time slots for a meeting.



A

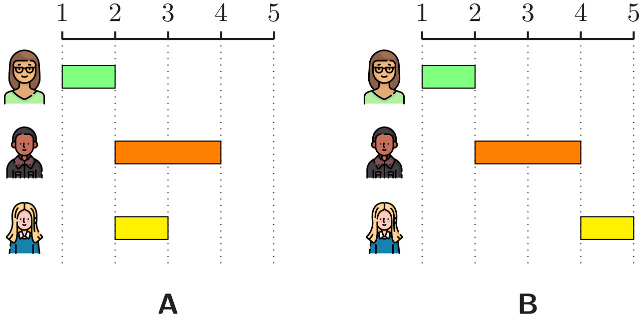


B

How should we aggregate the time slots?
Which kind of information should we elicit from the agents?

Choosing the time slot for a meeting

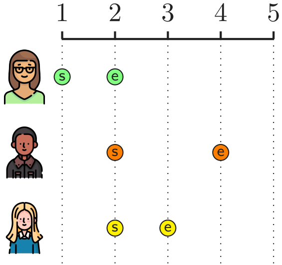
Option 0: Ask to submit the “whole” interval.



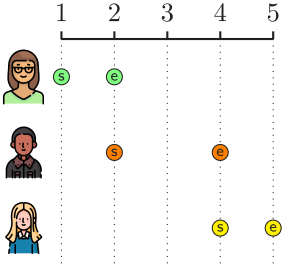
Aggregation of the time slots:
Intersection? Union? Plurality? ...

Choosing the time slot for a meeting

Option 1: Ask for the preferred starting time and ending time.



A



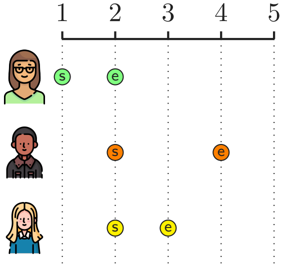
B

A: $s = (1, 2, 2)$, $e = (2, 4, 3)$

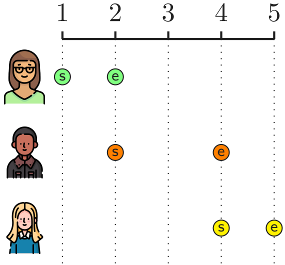
B: $s = (1, 2, 4)$, $e = (2, 4, 5)$

Choosing the time slot for a meeting

Option 1: Ask for the preferred starting time and ending time.



A



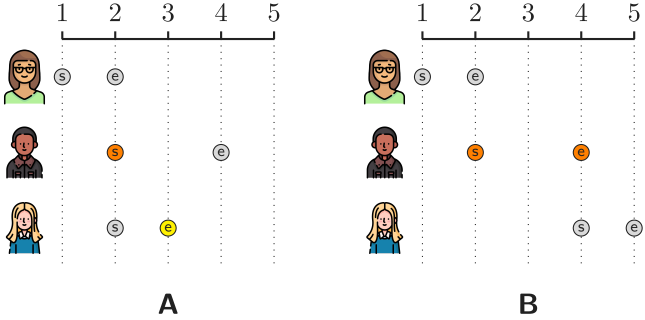
B

A: $s = (1, 2, 2)$, $e = (2, 4, 3)$ **B**: $s = (1, 2, 4)$, $e = (2, 4, 5)$

Aggregation: Use the median on the starting time and ending time.

Choosing the time slot for a meeting

Option 1: Ask for the preferred starting time and ending time.

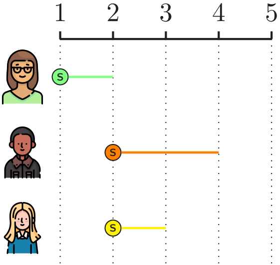


A: $[2, 3]$ **B**: $[2, 4]$

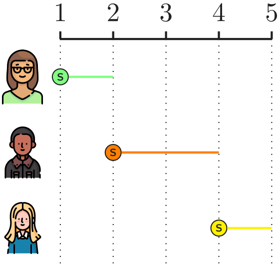
Aggregation: Use the median on the starting time and ending time.

Choosing the time slot for a meeting

Option 2: Ask for the preferred starting time and duration.



A



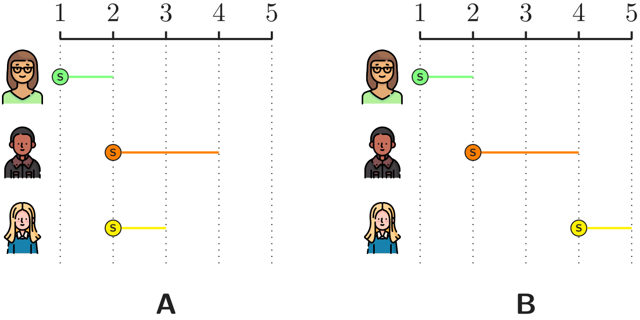
B

A: $s = (1, 2, 2)$, $d = (1, 2, 1)$

B: $s = (1, 2, 4)$, $d = (1, 2, 1)$

Choosing the time slot for a meeting

Option 2: Ask for the preferred starting time and duration.



A: $s = (1, 2, 2)$, $d = (1, 2, 1)$ **B**: $s = (1, 2, 4)$, $d = (1, 2, 1)$

Aggregation: Use the median on the starting time and . . . ?

Motivation

The example shows that, when using an aggregation rule for intervals that is defined in terms of *local* aggregators on the different components, the choice of representation matters.

- Does this happen *only* for the median-endpoint rule?
- Are there any 'good' rules *faithful* to multiple representations?
- Does the choice of the interval *scale* matter?

Talk overview

1. Formal framework
2. Impossibility results for discrete scales
3. Characterisation results for continuous scales
4. Conclusions and future work

Formal framework

Scales, intervals, and components

- A scale is a nonempty set $S \subseteq \mathbb{R}$ of real numbers with a minimum and maximum element.
  - $S = \{-3, 0, 2, 4, 7, 10, 12\}$ is a *discrete* scale
  - $S' = [a, b]$ for $a, b \in \mathbb{R}$ is a *continuous* scale
  - $S'' = [0, 1]$ is the *standard continuous* scale
- A (closed) interval is a nonempty subset $I \subseteq S$ with $\{z \in S \mid x < z < y\} \subseteq I$ for all $x, y \in I$ and $\inf(I), \sup(I) \in I$. The set of all (closed) intervals definable on S is denoted by $\mathcal{I}(S)$.
  - $I = \{0, 2, 4\}$ is an interval of S, while $\{4, 10\}$ is not
  - $I' = \{4\}$ is a *degenerate* interval of S
- A component is a function $\gamma : \mathcal{I}(S) \to D$, for a domain D.
  - left endpoint $\ell : I \mapsto \min(I)$
  - right endpoint $r : I \mapsto \max(I)$
  - width $w : I \mapsto \max(I) - \min(I)$

Faithful and unanimous rules

Fix a set $N = \{1, \ldots, n\}$ of agents. Each agent $a \in N$ submits an interval $I_a \in \mathcal{I}(S)$, which gives rise to a profile $\boldsymbol{I} = (I_1, \ldots, I_n)$.

- An aggregation rule $F : \mathcal{I}(S)^n \to \mathcal{I}(S)$ maps any such profile to a (collective) interval.
- A rule F is faithful to a component-representation $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_q)$, if there exists functions $f_k : D_k^n \to D_k$ for $k \in \{1, \ldots, q\}$ such that, for any $\boldsymbol{I}$, $F(\boldsymbol{I})$ is computed by using each f_k on the component-profile $(\gamma_k(I_1), \ldots, \gamma_k(I_n))$.
- A rule F is component-unanimous if $f_k(x, \ldots, x) = x$ for every $x \in D_k$ and for all $k \in \{1, \ldots, q\}$.

Basic results (1/3)

Question: Are there any rules that are faithful and unanimous for both the left-right and the left-width representation?

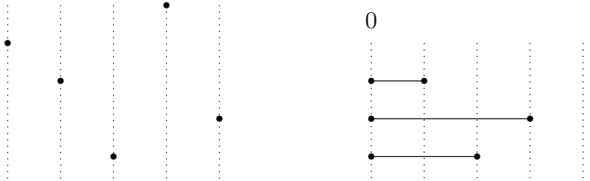
$\Rightarrow$ We start by proving some basic Lemmas (here: just a subset).

Lemma. Designing a rule that is both an (ℓ, r)-rule and an (ℓ, w)-rule is equivalent to designing an (ℓ, r, w)-rule.

Basic results (2/3)

Lemma. For the component-aggregators f_ℓ and f_r of any given (ℓ, r, w)-rule, it must be the case that $f_\ell = f_r$.

Lemma. For the component-aggregators f_ℓ, f_r and f_w of any given (ℓ, r, w)-rule defined on a scale S with $\min(S) = 0$, it must be the case that $f_\ell = f_r = f_{w\restriction S}$.



Basic results (3/3)

Lemma. For any given (ℓ, r)-rule F with $f_\ell = f_r$ that is defined on a scale S and any point profiles $\boldsymbol{x}, \boldsymbol{y} \in S^n$ with $\boldsymbol{x} \geq \boldsymbol{y}$, it is the case that $f(\boldsymbol{x}) \geq f(\boldsymbol{y})$ for $f := f_\ell = f_r$.

$\boldsymbol{y}$	$\dots$	$\boldsymbol{x}$
y_1	$\dots$	x_1
$\vdots$		$\vdots$
y_n	$\dots$	x_n
$f(\boldsymbol{y})$	$\dots$	$f(\boldsymbol{x})$

Impossibility results for discrete scales

The case for 2 agents

Lemma. For $n = 2$ and any given discrete scale S, every (ℓ, r, w)-rule is a dictatorship.

$\Rightarrow$ Same widths in input imply same widths in output, as (ℓ, w)-faithful.

x	x	0
x	y	$y - x$
x	$f(x,y)$	$f(x,y) - x$

y	y	0
x	y	$y - x$
$f(x,y)$	y	$y - f(x,y)$

The case for 2 agents

Lemma. For $n = 2$ and any given discrete scale S, every (ℓ, r, w)-rule is a dictatorship.

⇒ Same widths in input imply same widths in output, as (ℓ, w)-faithful.

$$\begin{array}{cc|c} x & x & 0 \\ x & y & y-x \\ \hline x & f(x,y) & f(x,y)-x \end{array} \qquad \begin{array}{cc|c} y & y & 0 \\ x & y & y-x \\ \hline f(x,y) & y & y-f(x,y) \end{array}$$

⇒ Transitivity of local dictatorships on points $(x, y), (y, z) \to (x, z)$.

$$\begin{array}{cc|c} x & x & 0 \\ y & z & z-y \\ \hline x & x & 0 \end{array} \qquad \begin{array}{cc|c} y & y & 0 \\ y & z & z-y \\ \hline y & y & 0 \end{array}$$

⇒ Induction on # of points in $[x, y]$: every (x, y) has (same) dictator.

Impossibility result for n agents

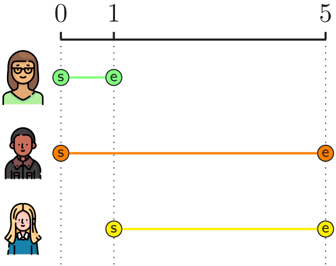
Theorem. For any given discrete scale S, every interval aggregation rule that is both an (ℓ, r)-rule and an (ℓ, w)-rule must be a dictatorship.

⇒ $\min(S) = 0$: follows from unanimity, the result for 2 agents, and the contrapositive of this Inductive Lemma:

- **Inductive Lemma.** For any given $n \geq 2$ and any given scale S with $\min(S) = 0$, if there exists a nondictatorial (ℓ, r, w)-rule for $n + 1$ agents, then also for n agents.

⇒ $\min(S) = b \neq 0$: construct a scale $S' = \{x - b \mid x \in S\}$ and proceed by contradiction (there is a non-dictatorial rule ...).

Special scale and restricted domain



What is special about this scale (no degenerate intervals)?

Characterisation results for continuous scales

Characterisation of weighted averaging rules

An (ℓ, r)-rule F is an (ℓ, r)-weighted averaging rule if there are constants $a_1, \ldots, a_n \in [0, 1]$ with $a_1 + \cdots + a_n = 1$ such that $f_\ell(\boldsymbol{x}) = f_r(\boldsymbol{x}) = a_1 \cdot x_1 + \cdots + a_n \cdot x_n$ for every $\boldsymbol{x} \in S^n$.

Theorem. For any continuous scale S, a continuous interval aggregation rule is both an (ℓ, r)-rule and an (ℓ, w)-rule if and only if it is an (ℓ, r)-weighted averaging rule.

- $\Rightarrow$ Use of *Cauchy's functional equation*: $f(x) + f(y) = f(x + y)$.
- $\Rightarrow$ Prove that for the scale $S = [0, 1]$ every continuous (ℓ, r, w)-rule is an (ℓ, r)-weighted averaging rule.
- ▶ Adding *anonymity*, we get $a_i = \frac{1}{n}$ for all $i \in N$.

Conclusions and future work

Summary and open questions

- The choice of representation of intervals heavily influences the aggregation rules we can design.
- Discrete scales: impossible to design nondictatorial rules that are faithful both to left-right and left-width representation.
- Continuous scales: weighted averaging rules are the only ones faithful both to left-right and left-width representation.

- What about *other representations*?
- What about *restricted domains*?
- What about similar questions in *other areas* of SCT?

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